

Exercise 8

Convert each of the following IVPs in 1–8 to an equivalent Volterra integral equation:

$$y^{(\text{iv})} + y'' + y = x, \quad y(0) = y'(0) = 1, \quad y''(0) = y'''(0) = 0$$

Solution

Let

$$y^{(\text{iv})}(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x .

$$\begin{aligned} \int_0^x y^{(\text{iv})}(t) dt &= \int_0^x u(t) dt \\ y'''(x) - y'''(0) &= \int_0^x u(t) dt \end{aligned}$$

Substitute $y'''(0) = 0$.

$$y'''(x) = \int_0^x u(t) dt \tag{2}$$

Integrate both sides again from 0 to x .

$$\begin{aligned} \int_0^x y'''(s) ds &= \int_0^x \left[\int_0^s u(t) dt \right] ds \\ y''(x) - y''(0) &= \int_0^x \int_0^s u(t) dt ds \end{aligned}$$

Substitute $y''(0) = 0$.

$$y''(x) = \int_0^x \int_0^s u(t) dt ds$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^s u(t) dt & dw &= ds \\ dv &= u(s) ds & w &= s \end{aligned}$$

and use the formula $\int v dw = vw - \int w dv$.

$$\begin{aligned} y''(x) &= s \int_0^s u(t) dt \Big|_0^x - \int_0^x su(s) ds \\ &= x \int_0^x u(t) dt - \int_0^x su(s) ds \\ &= x \int_0^x u(t) dt - \int_0^x tu(t) dt \\ &= \int_0^x (x-t)u(t) dt \end{aligned} \tag{3}$$

Integrate both sides again from 0 to x .

$$\int_0^x y''(r) dr = \int_0^x \left[\int_0^r (r-t)u(t) dt \right] dr$$

$$y'(x) - y'(0) = \int_0^x \int_0^r (r-t)u(t) dt dr$$

Substitute $y'(0) = 1$ and bring it to the right side.

$$y'(x) = 1 + \int_0^x \int_0^r (r-t)u(t) dt dr$$

In order to evaluate the double integral, switch the order of integration so that dr comes first.

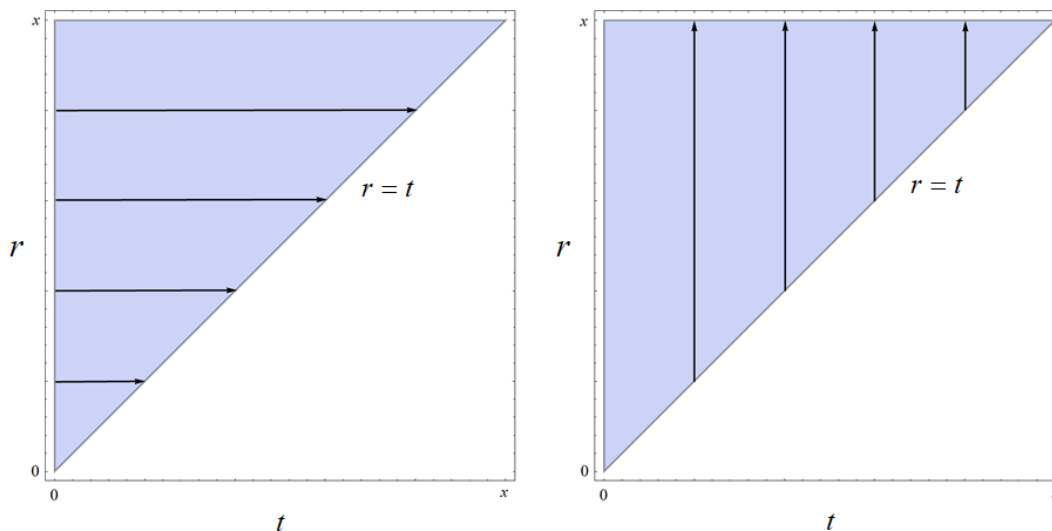


Figure 1: The current mode of integration in the tr -plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned} y'(x) &= 1 + \int_0^x \int_t^x (r-t)u(t) dr dt \\ &= 1 + \int_0^x \left[\frac{(r-t)^2}{2} \right]_t^x u(t) dt \\ &= 1 + \int_0^x \frac{(x-t)^2}{2} u(t) dt \\ &= 1 + \frac{1}{2} \int_0^x (x-t)^2 u(t) dt \end{aligned} \tag{4}$$

Integrate both sides again from 0 to x .

$$\int_0^x y'(q) dq = \int_0^x \left[1 + \frac{1}{2} \int_0^q (q-t)^2 u(t) dt \right] dq$$

$$y(x) - y(0) = x + \frac{1}{2} \int_0^x \int_0^q (q-t)^2 u(t) dt dq$$

Substitute $y(0) = 1$ and bring it to the right side.

$$y(x) = 1 + x + \frac{1}{2} \int_0^x \int_0^q (q-t)^2 u(t) dt dq$$

Switch the order of integration as we did before.

$$\begin{aligned} &= 1 + x + \frac{1}{2} \int_0^x \int_t^x (q-t)^2 u(t) dq dt \\ &= 1 + x + \frac{1}{2} \int_0^x \left[\frac{(q-t)^3}{3} \right]_t^x u(t) dt \\ &= 1 + x + \frac{1}{2} \int_0^x \left[\frac{(x-t)^3}{3} \right] u(t) dt \\ &= 1 + x + \frac{1}{6} \int_0^x (x-t)^3 u(t) dt \end{aligned} \tag{5}$$

Substitute equations (1), (2), (3), (4), and (5) into the original ODE.

$$y^{(iv)} + y'' + y = x \quad \rightarrow \quad u(x) + \left[\int_0^x (x-t)u(t) dt \right] + \left[1 + x + \frac{1}{6} \int_0^x (x-t)^3 u(t) dt \right] = x$$

Expand the left side.

$$\begin{aligned} u(x) + \int_0^x (x-t)u(t) dt + 1 + x + \frac{1}{6} \int_0^x (x-t)^3 u(t) dt &= x \\ u(x) + 1 + \int_0^x (x-t)u(t) dt + \int_0^x \frac{1}{6}(x-t)^3 u(t) dt &= 0 \\ u(x) + 1 + \int_0^x \left[(x-t) + \frac{1}{6}(x-t)^3 \right] u(t) dt &= 0 \\ u(x) + 1 + \int_0^x (x-t) \left[1 + \frac{1}{6}(x-t)^2 \right] u(t) dt &= 0 \end{aligned}$$

Therefore, the equivalent Volterra integral equation is

$$u(x) = -1 - \int_0^x (x-t) \left[1 + \frac{1}{6}(x-t)^2 \right] u(t) dt.$$